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THE SUPERSTAR PACKING PROBLEM

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Hell and Kirkpatrick proved that in an undirected graph, a maximum size packing by a set of non-singleton stars can be found in polynomial time if this star-set is of the form $\{S_1, S_2, \ldots, S_k\}$ for some $k \in \mathbb{Z}_+$ (S_i is the star with i leaves), and it is NP-hard otherwise. This may raise the question whether it is possible to enlarge a set of stars not of the form $\{S_1, S_2, \ldots, S_k\}$ by other non-star graphs to get a polynomially solvable graph packing problem. This paper shows such families of depth 2 trees. We show two approaches to this problem, a polynomial alternating forest algorithm, which implies a Berge–Tutte type min-max theorem, and a reduction to the degree constrained subgraph problem of Lovász.

1. Introduction

Let $\mathcal{F}$ be a set of undirected graphs. An *$\mathcal{F}$-packing* of a graph G is a subgraph Q of G, with the property that every component of Q is isomorphic to a member of $\mathcal{F}$. An $\mathcal{F}$-packing is called *maximum* if it covers a maximum number of vertices of G and it is an *$\mathcal{F}$-factor* if it covers every vertex of G. The *$\mathcal{F}$-packing problem* is to find a maximum $\mathcal{F}$-packing of G. Several authors (e.g. [2,3,5,6,9,10]) studied this family of problems. Polynomial time algorithms for finding maximum $\mathcal{F}$-packings and NP-completeness results have been given for various special sets $\mathcal{F}$ of graphs. The goal of the present paper is to introduce and solve a new polynomially solvable packing problem. This problem is much more involved than the other studied packing

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problems, which forecasts that when approaching the P–NP-completeness borderline one may face great difficulties in the field of graph packings.

Denote by S_i the *i-star*, that is the tree with i edges all adjacent to a vertex, called the *center*. We call the other i vertices the *leaves*. Hell and Kirkpatrick [6] proved that if $H \subseteq \mathbb{Z}_+ \setminus \{0\}$ then the $\{S_i : i \in H\}$-packing problem is polynomially solvable whenever H is of the form $\{1, 2, \ldots, k\}$, and it is NP-hard otherwise. One may raise the question: provided the problem is NP-hard, is it possible to add some other non-star graphs to $\mathcal{F} = \{S_i : i \in H\}$ to recover polynomiality? There are some known results in this flavor. Adding all trees with all degrees odd to $\{S_1, S_3, S_5, \ldots\}$ results in the $\mathcal{F}$-packing problem, where $\mathcal{F}$ consists of the graphs with all degrees odd. This problem is polynomial time solvable by an easy reduction to the matching problem. Another example is the NP-complete $\{S_k\}$-packing problem for $k \geq 2$. In this case the addition of all trees with highest degree exactly k results in the polynomially solvable k-piece packing problem [4]. Finally, if $H = \{l, l+1, \ldots, u\}$ and $l \geq 2$ then by adding to $\mathcal{F}$ all trees with highest degree between l and u we obtain another polynomially solvable packing problem [4].

In this paper we show that if $1 \in H$ and H has no two consecutive 'gaps', then for every $1 \leq b \in \mathbb{Z}_+$ a certain family of so-called 'superstars' (these are depth 2 trees) can be added to $\mathcal{F} = \{S_i : i \in H\}$ to recover polynomiality. (This will be called the '$\mathcal{C}_{H,b}$-packing problem'.) As H will be fixed throughout the paper, in the notations to follow we omit the reference to H.

Definition 1.1. An integer h is called a *gap* of $H \subseteq \mathbb{Z}_+$ if $h \notin H$ but H contains an integer bigger than h and an integer smaller than h. We say that H has *no two consecutive gaps* if $\min H \leq i \leq \max H$, $i \notin H$ implies $i+1 \in H$.

Definition 1.2. Let $H \subseteq \mathbb{Z}_+$. The star S_i is *forbidden* if i is a gap of H. For the integers $s \geq 0, t \geq 1$, a graph is said to be an (s,t)-*superstar* if it is constructed as follows: connect the center of an s-star to the centers of t forbidden stars. In a superstar every vertex inherits the notation *center* or *leaf*, except the center of the initial s-star, which is called the *supercenter* (or (s,t)-*supercenter*).

An $(s,1)$-superstar with $s \geq 1$ consists of two stars S_s and S_h with an edge connecting their centers, thus we also call it a *bi-star* $B_{s,h}$.

Finally we define the families of allowed graphs in our packings.

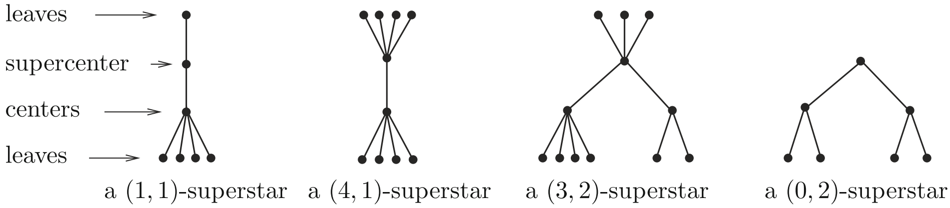


Figure 1. Examples of superstars, $H=\{1,3,5\}$

Definition 1.3. Let $H\subseteq\mathbb{Z}_+$ with no two consecutive gaps and $1=\min H$. Denote $u=\max H$. For $1\le b\in\mathbb{Z}_+$ let

$$\mathcal{C}_{H,b}=\{S_i:\ i\in H\}\cup\{(s,t)\text{-superstars: } 0\le s\le u,\ \ 1\le t\le b\}.$$

The requirement $1\in H$ ensures that $K_2\in\mathcal{C}_{H,b}$, a condition often yielding more easily tractable packing problems. Notice that if $H=\{1,\dots,u\}$ then $\mathcal{C}_{H,b}=\{S_i:\ 1\le i\le u\}$ independently of b, yielding the star packing problem of Las Vergnas [8]. Hence if $u=1$ we get the classical matching problem.

In the rest of the paper $H\subseteq\mathbb{Z}_+$ and $b\in\mathbb{Z}_+$ are fixed with $u=\max H\ge 2$, and they satisfy the properties in Definition 1.3.

By requiring $u\ge 2$ we exclude only the matching problem from our considerations. We do this for sake of convenience, since the approach to follow would be slightly more complicated if we wanted the matching problem to suit.

We present two approaches to the $\mathcal{C}_{H,b}$-packing problem. First we show a polynomial time alternating forest algorithm, which is a direct generalization of the classical matching algorithm of Edmonds. This algorithm needs an oracle deciding if a graph is ‘critical’ (defined later), which problem is not much easier than the original one. The algorithm implies a Berge–Tutte type min-max formula for the $\mathcal{C}_{H,b}$-packing problem. Then in Section 4 we show a direct reduction of the $\mathcal{C}_{H,b}$-packing problem to the degree constrained subgraph problem. This reduction uses the Gallai–Edmonds decomposition of the graph, and it yields an alternative algorithm for the $\mathcal{C}_{H,b}$-packing problem, and also the oracle needed for our alternating forest algorithm. This approach implies that the $\mathcal{C}_{H,b}$-packing problems are matroidal, calling an $\mathcal{F}$-packing problem *matroidal* if in every graph G the vertex sets coverable by $\mathcal{F}$-packings form the independent sets of a matroid. The proofs of some lemmas and the description of the alternating forest algorithm itself are contained in the Appendix.

Our Edmonds type algorithm has the peculiarity that the alternating forest may cover a vertex twice. Augmenting walks for improving b-matchings can also go through a vertex twice. However, for b-matchings, this twofold overlap is due to the restriction that no two consecutive gaps are allowed in a constraint, while in our case it comes from the two types of 'blades' in a superstar: leaves and forbidden stars. See [14] for packing algorithms where many-fold coverings appear.

There exist further tractable variants of the $\mathcal{C}_{H,b}$-packing problem. Let $H \subseteq \mathbb{Z}_+$ with no two consecutive gaps and $1 = \min H$. For $b, r \in \mathbb{Z}_+$ with $1 \le r \le b \le u+r$ let

$$\mathcal{S}_{H,b,r} = \{S_i \colon i \in H\} \cup \{(s,t)\text{-superstars} \colon 0 \le s \le u,\ 1 \le t \le b,\ s+t \le u+r\}.$$

In [14] it is shown that all considerations of the paper hold for the $\mathcal{S}_{H,b,r}$-packing problem (with small modifications in the alternating forest algorithm), implying a Berge–Tutte and even a Gallai–Edmonds type structure theorem to this family of packing problems. There exists yet another variant of the superstar packing problem, namely the case $r=0$, $1 \le b \le u-1$. Again, the considerations of the paper are valid, except the approach of Section 4, which is somewhat more difficult, see [7]. We do not treat these superstar packings here in order to bound the size of the paper. Though there surely exist NP-complete superstar packing problems too, we have not constructed such.

We mention also that along the same lines to follow, the superstar packing problems can be solved in their *local* version instead of the treated *global* version, that is if each vertex $v \in V(G)$ has an own constraint $H(v), b(v)$ satisfying the requirements of Definition 1.3.

Throughout the paper all graphs are simple and undirected. Let G be a graph and $U \subseteq V(G)$. $\Gamma_G(U)$ denotes the set of vertices $v \in V(G) - U$ which are adjacent to U in G, and $\delta_G(U)$ denotes the set of edges $e \in E(G)$ joining U to $G-U$. *Shrinking* U results in the graph G/U obtained from $G-U$ by joining a new vertex (called *shrunk* vertex) to each vertex of $\Gamma_G(U)$. We say that a subgraph Q of G *enters* U if $E(Q) \cap \delta_G(U) \neq \emptyset$. If $\mathcal{S}$ is a set of disjoint subgraphs of G then $\bigcup \mathcal{S} = \bigcup \{V(K) : K \in \mathcal{S}\}$. The number of connected components of G is denoted by $c(G)$. A graph F is called *factor-critical* if it has no perfect matching but $F - v$ has one for all $v \in V(F)$. $\mathbb{Z}_+$ denotes the set of non-negative integers.

2. Results

In this section we state our Berge–Tutte type min-max theorem for the $\mathcal{C}_{H,b}$-packing problem.

It is easy to see that every non-singleton factor-critical graph F has an $\{S_1, S_2\}$-factor. If $2 \notin H$ then, unless it is an odd circuit, F also has a

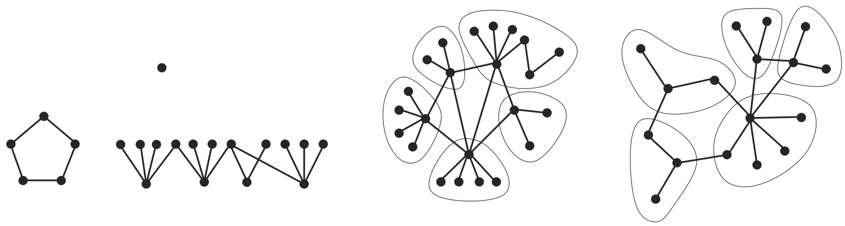
$\mathcal{C}_{H,b}$-factor with K_2-components and one bi-star $B_{1,2} \in \mathcal{C}_{H,b}$. Summarizing, the factor-critical graphs with no $\mathcal{C}_{H,b}$-factors are the singleton and, in case $2 \notin H$, the odd circuits. In the $\mathcal{C}_{H,b}$-packing problem the role of factor-critical graphs is played by graphs we call *hedgehogs*.

Definition 2.1. A nonempty tree is a *gap-tree* if, viewed as a bipartite graph with color classes A and D, the degree of each vertex in A is a gap of H. A connected graph W is a *small hedgehog* if either W is a gap-tree or W is an odd circuit and $2 \notin H$.

One may easily check by induction that a small hedgehog has no $\mathcal{C}_{H,b}$-factor.

Definition 2.2. A graph W is a *factor-critical union of small hedgehogs* $W_1, \ldots, W_{2l+1}$ if W_i is an induced subgraph of W for $1 \le i \le 2l+1$, $V(W_1), \ldots, V(W_{2l+1})$ partition $V(W)$, and shrinking every W_i results in a factor-critical graph, denoted by W/π. Here $\pi = \{W_1, \ldots, W_{2l+1}\}$ is a *decomposition* of W and W_i are the *small hedgehogs of* π.

Definition 2.3. A factor-critical union of small hedgehogs is a $\mathcal{C}_{H,b}$-*hedgehog* (or simply a *hedgehog*) if it has no $\mathcal{C}_{H,b}$-factor.



small hedgehogs, $H = \{1,3,5\}$ $\qquad$ $\mathcal{C}_{H,b}$-hedgehogs, $H = \{1,3,5\}$, $b = 1$

Figure 2. Examples of small hedgehogs and hedgehogs

Unfortunately, unlike in most known polynomially solvable packing problems, and similarly to the degree constrained subgraph problem, deciding if a factor-critical union of small hedgehogs has a $\mathcal{C}_{H,b}$-factor or not seems to be a difficult task. Only one method is known, which even finds a maximum $\mathcal{C}_{H,b}$-packing in any graph. This method is based on a reduction to the degree constrained subgraph problem, and worked out in Section 4. However, though polynomial solvability of the general problem indeed follows from this method, we do not know how to deduce the Berge–Tutte type theorem from it.

Definition 2.4. Let W be a hedgehog and $v \in V(W)$. If $W - v$ has a $\mathcal{C}_{H,b}$-factor then v is called *free*. If there exists a forbidden star subgraph S of W centered at v such that $W - V(S)$ has a $\mathcal{C}_{H,b}$-factor then v is called *fixed*.

Observe that in a small hedgehog all vertices of an odd circuit are both free and fixed, while in a gap-tree the vertices of A are fixed and not free, and the vertices of D are free and not fixed. We will show in Section 3 that each vertex of a hedgehog is either free or fixed or both.

Our main result is the following Berge–Tutte type theorem.

Definition 2.5. For $D \subseteq V(G)$ let $h_{H,b}(D)$ denote the number of $\mathcal{C}_{H,b}$-hedgehog components of $G[D]$. Furthermore, let $\Gamma^{fr}(D)$ (resp. $\Gamma^{fi}(D)$) denote those vertices in $V(G) - D$ which are adjacent to a free (resp. fixed) vertex of a $\mathcal{C}_{H,b}$-hedgehog component of $G[D]$. Finally, let $\mathrm{def}_{H,b}(G)$ denote the minimum number of vertices missed by a $\mathcal{C}_{H,b}$-packing of G.

Theorem 2.6. *If G is a simple undirected graph then*

$$\mathrm{def}_{H,b}(G) = \max_{D \subseteq V(G)} \left\{ h_{H,b}(D) - u\left|\Gamma^{fr}(D)\right| - b\left|\Gamma^{fi}(D)\right| \right\}.$$

3. Sketch of the proof

In this section we prove the Berge–Tutte type theorem 2.6. The proofs of most forthcoming lemmas are contained in the Appendix.

Definition 3.1. If W is a factor-critical union of small hedgehogs with decomposition π then a vertex $v \in V(W)$ is called π-*free* (resp. π-*fixed*) if it is free (resp. fixed) in its small hedgehog of π. $D_\pi \subseteq V(W)$ denotes the set of π-free vertices and let $A_\pi = V(W) - D_\pi$.

By the remark after Definition 2.4, every vertex is either π-free or π-fixed or both.

Definition 3.2. A decomposition π of a hedgehog W is *standard* if $vw \notin E(W)$ whenever v and w are π-free and contained in different small hedgehogs of π.

Note that if π is a standard decomposition of the factor-critical union W of small hedgehogs then each component of $W[D_\pi]$ is either a singleton or, in case $2 \notin H$, an odd circuit. There are examples that standard decompositions are not unique. By Corollary 3.11, standard decompositions have the feature that freeness and π-freeness coincide, and similarly for fixedness. In addition:

Lemma 3.3. *Every hedgehog has a standard decomposition.*

Proof. See page 43. ∎

The validity of the following two propositions are easy to check.

Proposition 3.4. *Connecting two graphs, both of which is either a forbidden star or a singleton, by an edge gives a graph with a $\mathcal{C}_{H,b}$-factor. Thus connecting two small hedgehogs by an edge gives a graph with a $\mathcal{C}_{H,b}$-factor. Thus connecting two hedgehogs by an edge gives a graph with a $\mathcal{C}_{H,b}$-factor.*

Proposition 3.5. *Let W be a hedgehog with decomposition π and let $v \in V(W)$. If v is π-free then $W-v$ has a $\mathcal{C}_{H,b}$-factor. If v is π-fixed then W has a forbidden star subgraph S centered at v such that $W-V(S)$ has a $\mathcal{C}_{H,b}$-factor. Both packings may be chosen such that every vertex in $D_\pi - v$, resp. $D_\pi - V(S)$, is either a leaf or the center of a K_2-component.*

It follows that every vertex of a hedgehog is either free or fixed or both. In every $\mathcal{C}_{H,b}$-packing Q of G we assume that for each vertex $v \in V(Q)$ it is determined whether v is a leaf, a center or the supercenter of its component P in Q. Ambiguity may occur in the following cases. If $P \simeq K_2$ then one specified vertex is the center and the other one is the leaf. If $P \simeq S_i$ with $i-1$ a gap of H then it is determined whether P is considered as an i-star or as a $(0,1)$-superstar with a distinguished vertex z of degree 1 as the supercenter. Remember that – abusing the notation – *we do not call such a vertex z a leaf* in the rest of the paper. Finally, if $P \simeq B_{i,j}$ where both i and j are gaps of H, it is determined which one of the two centers is the supercenter.

Definition 3.6. Let G be a graph and $D \subseteq V(G)$. A $\mathcal{C}_{H,b}$-packing Q of G is called *D-fit* if all leaves of all components of Q are contained in D.

Remember that in a D-fit packing a vertex of degree 1 may be allowed to belong to $V(G)-D$, provided it is the center of a K_2-component or the supercenter of a $(0,1)$-superstar.

Definition 3.7. For $D \subseteq V(G)$ let $D_c \subseteq D$ denote the set of vertices contained in a singleton or, in case $2 \notin H$, in an odd circuit component of $G[D]$.

The following two lemmas make it possible to enlarge a given $\mathcal{C}_{H,b}$-packing.

Lemma 3.8. *Let $D \subseteq V(G)$. If Q is a $\mathcal{C}_{H,b}$-packing of G then we can find in polynomial time a D_c-fit $\mathcal{C}_{H,b}$-packing entering every component of $G[D_c]$ entered by Q.*

Proof. See page 44. ∎

Lemma 3.9. *Assume that for some $D \subseteq V(G)$ every component of $G[D]$ is factor-critical, G has a matching M which matches $\Gamma_G(D)$ to the components of $G[D]$, and $G-(D \cup \Gamma_G(D))$ has a perfect matching. If Q is a D_c-fit $\mathcal{C}_{H,b}$-packing of G then we can find in polynomial time a $\mathcal{C}_{H,b}$-packing Q' of G which covers all vertices of G except at most one vertex in every component of $G[D_c]$ not entered by Q.*

Proof. See page 44. ∎

Next we prove Theorem 3.12, which is the 'easy' direction of our Berge–Tutte theorem 2.6. Along the way we prove that π-freeness and π-fixedness are independent of the choice of the standard decomposition π.

Lemma 3.10. *Let W be a hedgehog with a standard decomposition π. Let $\mathcal{S}$ be a set of vertex disjoint subgraphs of W consisting of singletons which are non-π-free and forbidden stars centered at non-π-fixed vertices. Then $W-\bigcup\mathcal{S}$ has no $\mathcal{C}_{H,b}$-factor.*

Proof. Assume $S \in \mathcal{S}$ is a forbidden star. Since no leaf v of S is π-free and $S-v$ is isomorphic to a star in $\mathcal{C}_{H,b}$, it is enough to prove the statement for the case when $\mathcal{S}$ contains only singletons. Let $T=\bigcup\mathcal{S}$ and assume that $W-T$ has a $\mathcal{C}_{H,b}$-factor. By definition, $T \subseteq A_\pi$. Recall that $W[D_\pi]$ consists of singleton and (in case $2 \notin H$) odd circuit components. Besides, $\Gamma_W(D_\pi)=A_\pi$, W has a matching M which matches A_π to distinct components of $W[D_\pi]$, and $W-(D_\pi \cup A_\pi)=\emptyset$ has a perfect matching. Hence it is possible to apply first Lemma 3.8 and then 3.9 to $D=D_\pi$ and to the $\mathcal{C}_{H,b}$-factor of $W-T$. What we get is a $\mathcal{C}_{H,b}$-factor of W, which is impossible. ∎

Corollary 3.11. *Let W be a hedgehog with a standard decomposition π. The vertex $v \in V(W)$ is π-free (π-fixed) if and only if it is free (fixed).*

Proof. By Proposition 3.5, if v is π-free (π-fixed) then it is free (fixed). The other direction follows from Lemma 3.10. ∎

Corollary 3.11 implies that all neighbors of a non-fixed vertex of a hedgehog are non-free.

Theorem 3.12. *If $Y \subseteq V(G)$, $\mathcal{S}$ is a set of hedgehog components of $G-Y$ and Q is a $\mathcal{C}_{H,b}$-packing of G, then at most $u|\Gamma^{fr}(\bigcup\mathcal{S})|+b|\Gamma^{fi}(\bigcup\mathcal{S})|$ components of $\mathcal{S}$ are covered by Q.*

Proof. Denote the components of $\mathcal{S}$ covered by Q by $\mathcal{S}'\subseteq\mathcal{S}$. We describe a mapping $\varphi:\mathcal{S}'\to Y$. A component $K\in\mathcal{S}'$ will be marked either *free* or *fixed*, with the property that if K is free (resp. fixed) then $\varphi(K)$ is adjacent to a free (resp. fixed) vertex of K. Let $K\in\mathcal{S}'$ and define $Q_K=Q[V(K)]$. If there exists a vertex $y\in Y$ for which $\deg_Q(y)=1$ and which is adjacent to some $u\in V(K)$ in Q then define $\varphi(K)=y$ and mark K according to u. So assume there are no such vertices $y\in Y$. It is clear that every component of Q_K is either a singleton or a forbidden star or has a $\mathcal{C}_{H,b}$-factor. Thus by virtue of Lemma 3.10 and Corollary 3.11, Q_K has a component S which is either a free singleton $\{v\}$ (Case 1), or a forbidden star subgraph of K with a fixed center v (Case 2).

Case 1: If v is a supercenter in component P of Q then let $\varphi(K)$ be a center of P, and if v is a leaf in Q then let $\varphi(K)$ be the neighbor of v in Q. Note that v cannot be a center because then all leaves of it would be contained in K by our assumption. Mark K free.

Case 2: If v is an (s,t)-supercenter in component P of Q then $S\simeq S_s$ and every center of P resides in Y. So choose $\varphi(K)$ one among them. If v is a center in a component P of Q then P must be a superstar and S consists of v and the leaves of v in P. Thus let $\varphi(K)$ be the supercenter of P. Mark K fixed.

This was the construction of φ. Let $y\in Y$ be adjacent to $\bigcup\mathcal{S}$ in G. If $\deg_Q(y)\le 1$ then $|\varphi^{-1}(y)|\le 1$. If y is a center in Q then $\varphi^{-1}(y)$ consists of at most u components in $\mathcal{S}'$ and at most one of them is fixed. Finally, if y is an (s,t)-supercenter in Q then $\varphi^{-1}(y)$ contains at most s free and at most t fixed components. Thus $|\mathcal{S}'|\le u|\Gamma^{fr}(\bigcup\mathcal{S})|+b|\Gamma^{fi}(\bigcup\mathcal{S})|$. ∎

Next we describe the specification of our $\mathcal{C}_{H,b}$-packing algorithm: the alternating forest it maintains, and the stopping criterion. The steps of the algorithm are detailed in the Appendix. The algorithm is mainly of theoretical relevance since it needs an oracle deciding if a factor-critical union of small hedgehogs has a $\mathcal{C}_{H,b}$-factor, which problem is not much easier than the original one. Indeed, only the method of Section 4 is known to work, which decides the existence of a $\mathcal{C}_{H,b}$-factor in general graphs. However, we do not know how to deduce Theorem 2.6 without this algorithm.

We say that the forest S with color classes Y_* and $\mathcal{S}\cup\mathcal{O}$ is an *alternating forest* if the following properties are satisfied, see Figure 3.

- Every vertex in Y_* is called either *free odd* or *fixed odd* (not both). There exists a function $\varphi: Y_*\to V(G)$ such that $\varphi^{-1}(v)$ contains at most one free odd and at most one fixed odd vertex for all $v\in V(G)$. Accordingly, a vertex in $\varphi(Y_*)=Y\subseteq V(G)$ can be *free odd*, *fixed odd* or both.
- $\mathcal{S}\cap\mathcal{O}=\emptyset$ and $\mathcal{S}\cup\mathcal{O}$ is a set of vertex disjoint hedgehog subgraphs of $G-Y$.

- The vertices in $\mathcal{O}$ have degree 1 in S, these are called *outside components*.
- In S, a free odd vertex $y \in Y_*$ is adjacent to exactly $u+1$ subgraphs in $\mathcal{S}$ and to at most b subgraphs in $\mathcal{O}$. If $yW \in E(S)$ for $W \in \mathcal{S}$ then $\varphi(y)$ is adjacent to a free vertex of W in G. If $yW \in E(S)$ for $W \in \mathcal{O}$ then W is a forbidden star subgraph of G and $\varphi(y)$ is adjacent to its center in G.
- In S, a fixed odd vertex $y \in Y_*$ is adjacent to exactly $b+1$ subgraphs in $\mathcal{S}$ and to at most u subgraphs in $\mathcal{O}$. If $yW \in E(S)$ for $W \in \mathcal{S}$ then $\varphi(y)$ is adjacent to a fixed vertex of W in G. If $yW \in E(S)$ for $W \in \mathcal{O}$ then W is a singleton subgraph of G adjacent to $\varphi(y)$ in G.
- If $|\varphi^{-1}(y)| = 2$ for a vertex $y \in Y$ then neither preimage of y is adjacent to outside components. (Such a vertex is 'covered by S twice'.)

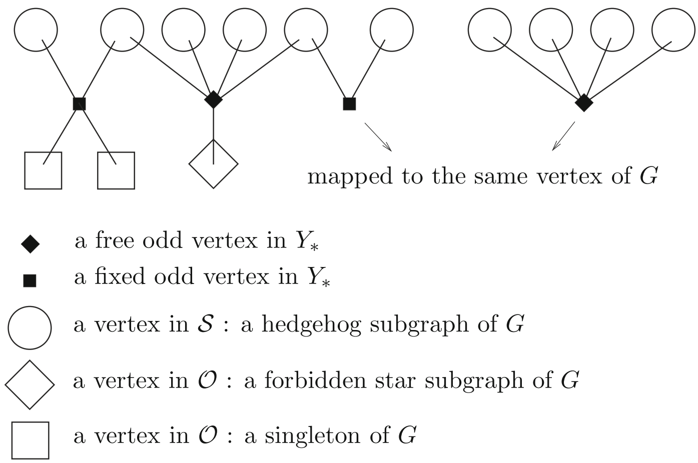


Figure 3. Two components of the alternating forest S, $u=3, b=1$

The algorithm maintains an alternating forest S with color classes Y_* and $\mathcal{S} \cup \mathcal{O}$, standard decompositions of the hedgehogs in $\mathcal{S}$ and a $\mathcal{C}_{H,b}$-factor Q of $G - (Y \cup \bigcup(\mathcal{S} \cup \mathcal{O}))$.

We describe an important operation on the alternating forest, called *blowup*. A vertex set $X \subseteq V(T)$ where T is a component of S is called *beautiful* if $T - X$ can be decomposed into vertex disjoint stars R of the following type: either R is centered at a free odd vertex y_R and R has exactly u leaves from $\mathcal{S}$ (and some from $\mathcal{O}$), or it is centered at a fixed odd vertex y_R and R has exactly b leaves from $\mathcal{S}$ (and some from $\mathcal{O}$). By Proposition 3.5 and Corollary 3.11 such a star R gives rise to a $\mathcal{C}_{H,b}$-factor Q_R of $G[\varphi(y_R) \cup \bigcup\{V(W) \in \mathcal{S} \cup \mathcal{O} : W \text{ is a leaf of } R\}]$. The union Q' of these Q_R's is a $\mathcal{C}_{H,b}$-packing of G, because the components of the Q_R's are disjoint, except if there exist two stars R_1 and R_2 with $\varphi(y_{R_1})=\varphi(y_{R_2})=y\in V(G)$. In this latter case Q_{R_i} has a u-star component centered at y and $Q_{R_{3-i}}$ has a $(0,b)$-superstar component supercentered at y (for some $i=1,2$). Thus the union Q' contains a (u,b)-superstar supercentered at y. *Blowing up* the beautiful $X\subseteq V(T)$ means deleting T from S, adding this $\mathcal{C}_{H,b}$-packing Q' to Q, and one more step is in order: if $\varphi^{-1}(y)=\{y^1,y^2\}$, $y^1\in Y_*\cap(V(T)-X)$ is free odd and $y^2\in Y_*\setminus V(T)$ is fixed odd then let the u-star component of Q' centered at y be P. Now delete P from Q' and add the leaves of P to $\mathcal{O}$ as singletons adjacent to y^2 in S. Do similarly if y^1 is fixed odd. At this level of generality we cannot specify what to do if $y^2\in X$ (something can always be done, see the Appendix).

In every step the algorithm either blows up a beautiful $X\subseteq V(T)$ for some component T of S (hence decreasing $c(S)$), or keeps $c(S)$ and increases $|E(S)|+\sum\{|E(W)|:W\in\mathcal{S}\cup\mathcal{O}\}$. It stops when every hedgehog of $\mathcal{S}$ is a connected component in $G-Y$ and if $y\in Y$ is not free odd (resp. not fixed odd) then y is adjacent to no free (resp. no fixed) vertices of hedgehogs of $\mathcal{S}$. As every step can be performed in polynomial time, and at most $|E(G)||V(G)|$ steps are possible, the algorithm is polynomial.

After termination, we define a $\mathcal{C}_{H,b}$-packing Q_{term} as follows. Note that if $W\in\mathcal{S}$ then $\{W\}$ is beautiful in its component in $\mathcal{S}$. Choose a hedgehog W_T in every component T of S, blow up the W_T's one by one, and add $\mathcal{C}_{H,b}$-factors of W_T-v to Q for a free vertex $v\in V(W_T)$. The resulting $\mathcal{C}_{H,b}$-packing Q_{term} misses $c(S)=h_{H,b}(\bigcup\mathcal{S})-u|\Gamma^{fr}(\bigcup\mathcal{S})|-b|\Gamma^{fi}(\bigcup\mathcal{S})|$ vertices, and so it is maximum by Theorem 3.12.

This is what the algorithm maintains, the how is described in the Appendix.

Proof of Theorem 2.6. After running the algorithm, the $\mathcal{C}_{H,b}$-packing Q_{term} misses $h_{H,b}(\bigcup\mathcal{S})-u|\Gamma^{fr}(\bigcup\mathcal{S})|-b|\Gamma^{fi}(\bigcup\mathcal{S})|$ vertices. The other direction follows from Theorem 3.12. ∎

4. A reduction to the degree constrained subgraph problem

In this section we show an alternative method which finds a maximum $\mathcal{C}_{H,b}$-packing in a graph in polynomial time. This method also serves as a realization of the oracle deciding if a factor-critical union of small hedgehogs has a $\mathcal{C}_{H,b}$-factor, making our alternating forest algorithm complete. From the previous results we use only Lemmas 3.8 and 3.9.

The *degree constrained subgraph problem* was introduced by Lovász [11]. Let G be an undirected graph with degree constraints $H(v)\subseteq\mathbb{Z}_+$ for all $v\in V(G)$. For a spanning subgraph F of G define $\delta_H^F(v)=\min\{|\deg_F(v)-h|:$

$h \in H(v)\}$ and let $\delta_H^F = \sum\{\delta_H^F(v) : v \in V(G)\}$. The minimum δ_H^F among the spanning subgraphs F is denoted by $\delta_H(G)$. A subgraph F is called H-*optimal* if $\delta_H^F = \delta_H(G)$ and it is an H-*factor* if $\delta_H^F = 0$, that is if $\deg_F(v) \in H(v)$ for all vertices $v \in V(G)$. The *degree constrained subgraph problem* is to determine the value of $\delta_H(G)$. Lovász [12] developed a structure theory to the degree constrained subgraph problem in case $H(v)$ is forbidden to have two consecutive gaps for every $v \in V(G)$. He proved that the problem is NP-complete without this restriction. Later, Cornuéjols [1] proved that the degree constrained subgraph problem is polynomial time solvable in case $H(v)$ has no two consecutive gaps for all $v \in V(G)$.

In this section $V(G) = D \dot\cup A \dot\cup C$ denotes the classical Gallai–Edmonds decomposition of G. By the Gallai–Edmonds structure theorem, every component of $G[D]$ is factor-critical, G has a matching M which matches $A = \Gamma_G(D)$ to the components of $G[D]$, and $G-(D\cup A)$ has a perfect matching. Thus Lemmas 3.8 and 3.9 clearly imply that it is enough to prove the following lemma.

Theorem 4.1. *One can find in polynomial time a $\mathcal{C}_{H,b}$-packing entering a maximum number of components in $G[D_c]$.*

Proof. We make a reduction to the degree constrained subgraph problem as follows. Each component of $G[D_c]$ is replaced by a gadget shown in Figure 4, each vertex of A is replaced by a gadget shown in Figure 5, let $H(v) = \{0, 1, \ldots, b\}$ for every $v \in V(G) - D_c - A$ and finally, delete $E(G - D_c - A)$. Note that we completely ignore the edges induced by $V(G)-A$. Let us denote the resulting graph by G^* with degree constraint H, and let $X = \{x_K : K$ is a component of $G[D_c]\}$.

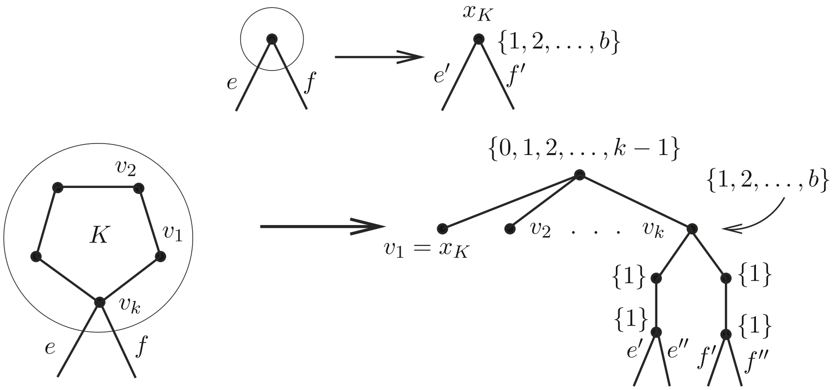


Figure 4. Gadgets replacing a singleton and an odd circuit component K of $G[D_c]$

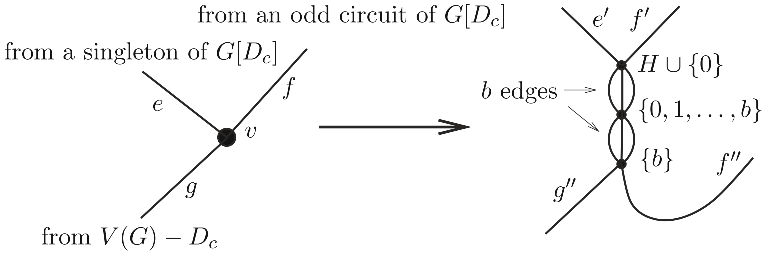


Figure 5. A gadget replacing $v \in A$

Denote the minimum number of components of $G[D_c]$ not entered by a $\mathcal{C}_{H,b}$-packing by d. We prove that $\delta_H(G^*) = d$, and that from an H-optimal subgraph of G^* one can construct in polynomial time a $\mathcal{C}_{H,b}$-packing of G entering all but d components of $G[D_c]$, and vice versa.

Let F be an H-optimal spanning subgraph of G^*. It is not hard to check that we can assume that $\deg_F(v) \in H(v)$ for all vertices $v \in V(G^*) - X$ and $\deg_F(x) \leq b$ for all $x \in X$. Define $\hat{E} \subseteq E(G)$ as follows. Let $e \in \hat{E}$ if $e' \in E(F)$ (an *upper* edge) and let $e \in \hat{E}$ if $e'' \in E(F)$ (a *lower* edge). Observe that at least one end vertex of every edge in $\hat{E}$ resides in A, and that $\hat{E}$ enters a component K of $G[D_c]$ if and only if $\deg_F(x_K) \in H(x_K)$. In addition, $\hat{E}$ satisfies the next property.

(1) For $v \in A$ let s (resp. t) denote the number of upper (resp. lower) edges of $\hat{E}$ incident to v. Now $s \leq u$, $t \leq b$ and if $t = 0$ then $\deg_{\hat{E}}(v) \in H \cup \{0\}$.

Call an edge set $E' \subseteq \hat{E}$ *good* if it satisfies Property (1) and it enters all components of $G[D_c]$ entered by $\hat{E}$. Call $e \in E'$ *dangerous* if $E' - e$ enters less components of $G[D_c]$ then E'. Now we describe a procedure finding a minimal good edge set E'. In the beginning let $E' = \hat{E}$.

Step 1. While there is an edge $e \in E'$ such that $E' - e$ is good: delete e from E'. Go to step 2.

In the moment we get stuck every non-dangerous edge e has the property that for an end vertex $c_e \in A$ it holds that $\deg_{E'-e}(c_e)$ is a gap of H, and $\delta_{E'-e}(c_e)$ contains only upper edges.

Step 2. If there is a non-dangerous edge e such that $\delta_{E'-e}(c_e)$ contains a non-dangerous edge f then delete $\{e, f\}$ from E'. E' is good unless there exists an odd circuit component K of $G[D_c]$ such that $\delta_{E'}(V(K)) = \{e, f\}$. So in this case instead of deleting $\{e, f\}$, delete f from E' and designate e to be a lower edge. Go to step 1. Otherwise stop.

In the end of this procedure, all non-dangerous edges $e = c_e z_e$ have the property that $c_e \in A$, $\deg_{E'-e}(c_e)$ is a gap of H and $\delta_{E'-e}(c_e)$ contains only upper *dangerous* edges. Observe that if $z_e \notin A$ then $\deg_{E'}(z_e) \leq b$ and the edges in $\delta_{E'}(z_e)$ are all non-dangerous. Besides, if $z_e \in A$ then $\delta_{E'}(z_e)$ contains only (upper or lower) *dangerous* edges going to D_c and lower non-dangerous edges going to A. If $e \in E'$ is a dangerous lower edge joining $v \in V(K)$ to A for an odd circuit component K of $G[D_c]$ then add the two incident edges of v in K to E'. All in all, the resulting E' has the property that if a component P of $(V(G), E')$ contains a non-dangerous edge $c_e z_e$ then P is an allowed $\mathcal{C}_{H,b}$-superstar with supercenter z_e, and if P contains only dangerous edges then it is isomorphic to a member of $\mathcal{C}_{H,b}$ and it has supercenter or center in A. Hence we have a D_c-fit $\mathcal{C}_{H,b}$-packing of G which enters every component K of $G[D_c]$ which was entered by $\hat{E}$, that is for which $\deg_F(x_K) \in H(x_K)$. So $d \leq \delta_H(G^*)$.

On the other hand, if Q is a $\mathcal{C}_{H,b}$-packing of G entering a maximum number of components of $G[D_c]$ then Lemma 3.8 gives a D_c-fit $\mathcal{C}_{H,b}$-packing Q' entering every component of $G[D_c]$ entered by Q. Define $\hat{E} \subseteq E(G^*)$ as follows. For $e = vw \in E(Q')$ with $w \in A$ define $e'' \in \hat{E}$, except if v is a leaf in Q' in which case let $e' \in \hat{E}$. Now it is easy to check that $\hat{E}$ can be extended to a spanning subgraph F of G^* such that $\deg_F(v) \notin H(v)$ only if $v = x_K$ for a component K of $G[D_c]$ not entered by Q', in which case $\deg_F(x_K) = 0$. Thus $\delta_H^F \leq d$. ∎

These considerations yield the matroidal property of the $\mathcal{C}_{H,b}$-packing problem. We use the following statement [13]. A subgraph F of G is called an *H-subgraph* if $\deg_F(v) \in H(v)$ for all $v \in V(F)$.

Theorem 4.2 (**[13]**). *Let G be a graph and let $1 \in H(v)$ be a constraint with no two consecutive gaps for all $v \in V(G)$. Then those vertex sets which can be covered by H-subgraphs form a matroid on $V(G)$.*

Theorem 4.3. *The vertex sets of G which can be covered by a $\mathcal{C}_{H,b}$-packing form a matroid.*

Proof. Note that $1 \in H(y)$ for all $y \in V(G^*)$ except for one vertex y_v in the gadget of every $v \in A$, where $H(y_v) = \{b\}$. For such vertices subdivide every edge of G^* incident to y_v by two new vertices with constraints $\{1\}$, and then replace y_v by b distinct copies with the same set of neighbors as y_v. The resulting graph is G', and let the constraints be $\{1\}$ on these b vertices. Now we can apply Theorem 4.2 to G'. Denote by M the matroid consisting of the vertex sets covered by H-subgraphs of G'. Contract $V(G')-$

X in M. Now take a series extension of x_K on $V(K)$ for each component K of $G[D_c]$. Finally, direct sum the elements of $V(G){-}D_c$ as bridges. The above considerations imply that the independent sets of the resulting matroid are exactly the vertex sets which can be covered by a $\mathcal{C}_{H,b}$-packing of G. ∎

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Appendix

A.1. Proofs

Whenever we make operations on a $\mathcal{C}_{H,b}$-packing we apply *cuttings*.

Definition A.1. 1. *Cutting* a leaf v in a component $P \simeq S_i$ of Q means deleting v if $i-1 \in H$. If $i-1 \notin H$ then simply designate v to be the supercenter (to the center if $i=1$).

2. *Cutting* a leaf v in an $(s,1)$-superstar with $s \geq 1$ (a bi-star) means deleting v, and in case v was connected to the center c, deleting the edge joining c to the supercenter. There is one exception: if v is connected to the center c and s is a gap of H then delete v, and designate c to be the supercenter.
3. *Cutting* a leaf v in an (s,t)-superstar with $t \geq 2$ means deleting v, and in case v was connected to a center c, deleting the edge joining c to the supercenter.

   Cutting a leaf edge means cutting its leaf vertex.

Observe that if v is a leaf in a component P of a $\mathcal{C}_{H,b}$-packing Q and a neighbor u of v is missed by Q then cutting v in P add joining u to v results in a $\mathcal{C}_{H,b}$-packing, unless $2 \notin H$ and $P \simeq K_2$.

Lemma A.2. *Let W be a factor-critical union of small hedgehogs with decomposition π and v_1, v_2 be two adjacent vertices which are π-free, in two distinct small hedgehogs of π. Then one can find either a $\mathcal{C}_{H,b}$-factor of W or a standard decomposition of W in linear time.*

Proof. For $i=1,2$, let Q_i be a $\mathcal{C}_{H,b}$-factor of $W - v_i$ guaranteed by Proposition 3.5. Q_i can clearly be chosen such that every small hedgehog is entered by at most one edge of Q_i. If v_{3-i} is the center of a K_2-component P of Q_i then swap the leaf and the center of P. So we assume that v_{3-i} is a leaf in Q_i. Cut v_{3-i} and join v_i to v_{3-i} in Q_i. In this way we get a $\mathcal{C}_{H,b}$-packing unless $2 \notin H$ and Q_i covers v_{3-i} by a copy of K_2. So we assume this for $i=1,2$. Consider a path R of maximal length starting at v_1 and containing alternately edges of $E(Q_2) \setminus E(Q_1)$ and $E(Q_1) \setminus E(Q_2)$ leading to vertices of degree 1 in Q_2 and Q_1, respectively. Algorithmically, if we traverse along such a walk R edge by edge, then we cannot cover a vertex more than once so R is really a path, which ends when the last vertex has no neighbor in Q_i of degree 1.

If R has odd length then its last vertex z is either a supercenter in a $(0,t)$-superstar or a leaf in a non-K_2 component P of Q_1. So Q_1 can be augmented to a $\mathcal{C}_{H,b}$-factor of W by cutting z in P if it was a leaf and swapping edges and non-edges along R.

If R has even length then its last vertex z is either a supercenter in a $(0,t)$-superstar or a leaf in a non-K_2 component P of Q_2 or $z = v_2$. In the first two cases Q_2 can be augmented to a $\mathcal{C}_{H,b}$-factor by cutting z in P if it was a leaf and swapping edges and non-edges along $v_2v_1 + R$.

Hence we can assume $z = v_2$ and $C = R + v_1v_2$ is an odd circuit. By starting at v_2, the above considerations imply that all components of Q_i intersecting C are copies of K_2 for $i=1,2$. Suppose that there exists a small hedgehog W' of π intersecting both C and $W-V(C)$. We may assume that $E(C)\cap E(Q_1)$ has an edge xy with $x\notin V(W')$, $y\in V(W')$. If W' is an odd circuit then – as W' is entered by at most one edge of $Q_i - C$ traverses an even length section of it then leaves W' either on an edge of Q_2 or on v_1v_2. Assume W' is a gap-tree. Q_1 enters W' so y is free in W' since otherwise it could not be contained in a K_2-component of Q_1. It is not hard to see that C goes along an even length path subgraph of W' traversing free vertices and degree 2 fixed vertices alternately, and C leaves W' either on an edge of Q_2 or on v_1v_2. In both cases, Q_1 has an s-star component S with $s\le u-2$ such that $V(S)\subseteq V(W')\setminus V(C)$ and the center of S is adjacent to C. Now $Q_1-V(S)-V(C)$ is a $\mathcal{C}_{H,b}$-factor of $W-V(S)-V(C)$. Moreover, $W[V(S)\cup V(C)]$ has a $\mathcal{C}_{H,b}$-factor with K_2-components and one bi-star $B_{s,2}$ (recall that $2\notin H$ and hence it is a gap of H).

If C is not an induced subgraph of W then $W[V(C)]$ has a $\{K_2,B_{1,2}\}$-factor. Otherwise every hedgehog $W'\in\pi$ for which $V(W')\subseteq V(C)$ is a path of even length (possibly 0). Let the subgraph of W/π induced by these hedgehogs be ϱ. Packing Q_1 gives rise to a perfect matching of $W/\pi-V(\varrho)$ showing that ϱ is a nice odd circuit in W/π. Thus $(W/\pi)/V(\varrho)$ is factor-critical. Hence declare C to be a new small hedgehog in this new factor-critical decomposition. Iterate the above procedure until the decomposition is standard. ∎

Lemma A.3. *If $u\neq v$ are two non-adjacent vertices of a small hedgehog W then either $W+uv$ has a $\mathcal{C}_{H,b}$-factor (which can be found in linear time) or $W+uv$ is a factor-critical union of small hedgehogs (and a standard decomposition of it can be presented in polynomial time).*

Proof. If W is an odd circuit then $W+uv$ has a $\{K_2,B_{1,2}\}$-factor. If W is a gap-tree then $W+uv$ contains a circuit C. Let $E'=E(C)$ if both u and v are free or both are fixed and let $E'=E(C)-uv$ otherwise. The graph $W'=(W+uv)-E'$ has an odd number of components, each of them either is a small hedgehog or has a $\mathcal{C}_{H,b}$-factor. It is not hard to check that if a component of W' has a $\mathcal{C}_{H,b}$-factor then also $W+uv$ has one. Otherwise $W+uv$ is a factor-critical union of small hedgehogs. A standard decomposition of it can be found by Lemma A.2. ∎

Lemma 3.3. *Every hedgehog has a standard decomposition.*

Proof. Follows from Lemmas A.2 and A.3. ∎

Lemma 3.8. Let $D \subseteq V(G)$. If Q is a $\mathcal{C}_{H,b}$-packing of G then we can find in polynomial time a D_c-fit $\mathcal{C}_{H,b}$-packing entering every component of $G[D_c]$ entered by Q.

Proof. First delete all components of Q which are disjoint from D_c. Then do the following cuttings until we get a D_c-fit $\mathcal{C}_{H,b}$-packing. Let $v \notin D_c$ be a leaf in a component P of Q. If $P \simeq S_i$, $i-1 \notin H$ and P has one more vertex $w \notin D_c$ of degree 1 (possibly the supercenter) then delete both v and w from Q. Otherwise simply cut v in P. In this way we disconnected a component K of $G[D_c]$ from $\Gamma_G(D_c)$ only if $2 \notin H$, K is an odd circuit, and

1. either $P \simeq S_3$ with center in K and degree 1 vertices $x \in V(K)$, $v, w \in \Gamma_G(D_c)$ (and v, w were deleted),
2. or P is a superstar and a center $c \in V(K)$ of P has exactly 2 adjacent leaves: $x \in V(K)$ and $v \in \Gamma_G(D_c)$ (and v was cut),
3. or P is an $(s,1)$-superstar and its supercenter z together with its one leaf are contained in K (a leaf of the center was cut).

In case 1., delete x and v from Q, in case 2., cut x and in case 3., cut the leaf neighbor of z. ∎

Lemma 3.9. *Assume that for some $D \subseteq V(G)$ every component of $G[D]$ is factor-critical, G has a matching M which matches $\Gamma_G(D)$ to the components of $G[D]$, and $G-(D \cup \Gamma_G(D))$ has a perfect matching. If Q is a D_c-fit $\mathcal{C}_{H,b}$-packing of G then we can find in polynomial time a $\mathcal{C}_{H,b}$-packing Q' of G which covers all vertices of G except at most one vertex in every component of $G[D_c]$ not entered by Q.*

Proof. Delete all components of Q which do not meet both D and $\Gamma_G(D)$.

We need to extend the operation of cutting. In the case $2 \notin H$ call an edge *bad* if it joins the supercenter z of an (s,t)-superstar P to a center c of P with $\deg_P(c)=3$ such that c together with both of its leaves u_1, u_2 are contained in an odd circuit component K of $G[D_c]$. *Cutting* the bad edge cz in P means to delete c, u_1 and u_2. However, this is not allowed if $t=1$ and $s \notin H$. So if $s=0$ then delete the whole P, while if s is a gap of H then delete u_1 and u_2 and designate c to be the supercenter.

If a component P of Q is a superstar with supercenter $z \in D_c$ then delete all leaves adjacent to z from Q. Call component K of $G[D]$ *pretty* if either K contains only $(0,t)$-supercenters of Q, or at most one edge of Q enters K which is either a leaf or bad.

Do the following procedure while there exists a component K of $G[D]$ which is not pretty. Observe that K is an odd circuit in D_c. Now $|E(Q) \cap$

$\delta_G(V(K))| \geq 2$ and there exists an edge $f = uv \in E(Q)$ with $u \in \Gamma_G(D)$, $v \in V(K)$ such that either v is a leaf in Q or f is bad. Let P be the component of Q containing f. Cut f – provided this does not disconnect any component of $G[D_c]$ from $\Gamma_G(D)$ in Q. Now either $f \notin E(Q)$ any more, or $P \simeq S_i$ is centered at u with $i-1 \notin H$, and v becomes the supercenter of P. In this latter case if P has two degree 1 vertices in D_c such that deleting both of them no component of $G[D_c]$ disconnects from $\Gamma_G(D)$ then delete them from Q; and if $E(Q) \cap \delta_G(V(K)) = \{vu, wu\}$, then add to $P - w$ the two neighbors of v in K such that uv will be a bad edge in a new bi-star $B_{2,i-2}$. It is not hard to see that this procedure is finite, and in the end every component of $G[D]$ is pretty.

Now we cover $\Gamma_G(D)$. Do the next procedure until there is a vertex $v \in \Gamma_G(D) \setminus V(Q)$. Let $e \in M$ be the edge joining v to u residing in component K of $G[D]$. If K is joined by only one edge $f \in E(Q)$ to $\Gamma_G(D)$ which is either a leaf or bad then cut f. Now add e to Q. The addition of e either creates a K_2-component of Q or adjoins v as a leaf to a $(0,t)$-supercenter. In this way it is possible that we spoil the prettiness of a component K' of $G[D_c]$. This can happen if f is a leaf of a $(0,1)$-superstar $P \simeq S_i$ of Q with supercenter $z \in V(K')$, and K' is entered by at least two edges of Q. In this case after the cutting of f the new supercenter is u, so delete f and z from Q. $M \cap E(Q)$ strictly increases at all of these steps so finally $\Gamma_G(D)$ will be covered by Q.

If K is a component of $G[D_c]$ not entered by the new Q then add to Q a perfect matching of $K - v$ for some $v \in V(K)$. If K is a component of $G[D - D_c]$ not entered by Q then add to Q a $\mathcal{C}_{H,b}$-factor of K. Let K be a component of $G[D]$ entered by Q. There are three possibilities by the prettiness of K.

1. All vertices in $V(K) \cap V(Q)$ are $(0,t)$-supercenters and at most one $(1,t)$-supercenter (created during the covering of $\Gamma_G(D)$). Choose $v \in V(K)$ to be the $(1,t)$-supercenter (if any).
2. Only one vertex of K is covered by Q, a leaf v.
3. K is entered by only one edge of Q, which is a bad edge with center $c \in V(K)$ and leaves $u_1, u_2 \in V(K)$. Choose $v = u_1$.

In all three cases take the union of a perfect matching of $K - v$ and Q, moreover, in case 1., delete those edges of this matching which join the supercenters of two superstars. At this point $G - (D \cup \Gamma_G(D))$ contains only $(0,t)$-supercenters. So finally, add a perfect matching N of $G - (D \cup \Gamma_G(D))$ to Q and delete those edges of N which join the supercenters of two superstars. This is the $\mathcal{C}_{H,b}$-packing Q'. ∎

A.2. The alternating forest algorithm

What the algorithm maintains is specified in page 35. Throughout we make use of an oracle deciding if a factor-critical union of small hedgehogs has a $\mathcal{C}_{H,b}$-factor.

Algorithm for the $\mathcal{C}_{H,b}$-packing problem

Step 1. If Q is a $\mathcal{C}_{H,b}$-factor of G, stop. Otherwise let $\mathcal{S}$ consist of the singletons in $V(G)-V(Q)$ and let $E(S)=Y_*=\mathcal{O}=\emptyset$. Go to step 2.

Step 2. Look for an edge $xz\in E(G)$, not the edge of a hedgehog of $\mathcal{S}$, such that either x is free in a hedgehog W of $\mathcal{S}$ and z is not a free odd vertex, or x is fixed in a hedgehog W of $\mathcal{S}$ and z is not a fixed odd vertex. If no such edge exists, stop. Otherwise denote the component of S containing W by T and distinguish the following cases. Continue at step 2 in the end of each case.

Case 1. $z\in V(W)$. Using Lemmas A.2, A.3 and the oracle deciding if a factor-critical union of small hedgehogs is a hedgehog, we can either obtain a $\mathcal{C}_{H,b}$-factor of $W+xz$ or conclude that $W+xz$ is a hedgehog and obtain a standard decomposition of it. If $W+xz$ has a $\mathcal{C}_{H,b}$-factor Q' then blow up W and add Q' to Q. Otherwise note that the vertices free (fixed) in W are also free (fixed) in $W+xz$. We grow S by adding xz.

Case 2. z is in a hedgehog $W^z\neq W$ of T. Adding the edge W^zW creates a circuit C in S. Let $X_C\subseteq V(T)$ consist of $Y_*\cap V(C)$ and of all hedgehogs in $\mathcal{S}\cup\mathcal{O}$ adjacent in S to $Y_*\cap V(C)$. Note that X_C is beautiful. For $y\in Y_*\cap V(C)$ denote $\mathcal{S}_y\subseteq\mathcal{S}\cup\mathcal{O}$ the set of neighbors of y in S, not on C. Observe that $X_y:=\{y\}\cup\mathcal{S}_y\cup\{W,W^z\}$ is beautiful.

Assume that we have $\varphi(y^1)=\varphi(y^2)=y\in V(G)$ for $y^1\in Y_*\cap V(C)$ and $y^2\in Y_*$. ($y^2\in V(C)$ is also possible.) Assume first that y^1 is free odd. Blow X_{y^1} up. Then for all $W\in\mathcal{S}_{y^1}$ choose a vertex v_W free in W and adjacent to y in G. Add $\mathcal{C}_{H,b}$-factors of $G[V(W)\cup V(W^z)]$ and of $W-v_W$ for $W\in\mathcal{S}_{y^1}$ to Q. Finally, we distinguish two cases. If $y^2\in V(T)$ then the new Q contains a $(0,b)$-superstar P supercentered at y^2. Now join the $u-1$ vertices v_W to y as leaves of P. One the other hand, if $y^2\notin V(T)$ then add the $u-1$ vertices v_W as singletons to $\mathcal{O}$ and join them to y^2 in S.

Proceed in a similar manner if y^1 is fixed odd and y^2 is free odd. So we assume that if $y\in Y$ is both free odd and fixed odd then its preimages avoid C. Thus we can define a subgraph G_C of G induced by $Y_*\cap V(C)$ and $\bigcup\{V(W):W\in X_C\cap(\mathcal{S}\cup\mathcal{O})\}$.

If $G[\varphi(y)\cup\bigcup\mathcal{S}_y]$ has a $\mathcal{C}_{H,b}$-factor Q_y for some $y\in Y_*\cap V(C)$ then blow up X_y and add Q_y and a $\mathcal{C}_{H,b}$-factor of $G[V(W)\cup V(W^z)]$ to Q. All in all, we are left with the case that if $y\in Y_*\cap V(C)$ is free odd then $\mathcal{S}_y\cap\mathcal{O}=\emptyset$ and $\varphi(y)$ is connected in G to only non-fixed vertices in the $u-1$ hedgehogs in $\mathcal{S}_y$ with $u-1\notin H$; besides, if $y\in Y_*\cap V(C)$ is fixed odd then $b=1$ and $\mathcal{S}_y\subseteq\mathcal{O}$ have size $i\notin H$. So G_C is a factor-critical union of small hedgehogs. Take a standard decomposition of G_C by Lemmas A.2 and A.3. Note that the free (fixed) vertices of hedgehogs of G_C-Y-xz are free (fixed) in G_C as well. Invoking the oracle, decide if G_C has a $\mathcal{C}_{H,b}$-factor Q'. If yes then add Q' to Q and blow up X_C. Otherwise delete X_C from $\mathcal{S}\cup\mathcal{O}$ and Y_*, add G_C to $\mathcal{S}$ and join G_C to a vertex $y\in Y_*$ in S if $y\in Y_*\setminus X_C$ was adjacent to X_C in the old S. The construction guarantees that we get a new alternating forest.

Case 3. z is in a hedgehog $W^z\in\mathcal{S}$ in component $T^z\neq T$ of S. Blow up W, then blow up W^z and add a $\mathcal{C}_{H,b}$-factor of $G[V(W)\cup V(W^z)]$ to Q.

Case 4a. x is free in W and $z\in Y$ is a non-free odd vertex. Assume $\varphi(z^1)=z$. ($Wz^1\in E(S)$ is possible!) Denote by $\mathcal{O}_z\subseteq\mathcal{O}$ the set of outside singletons adjacent to z^1 in S. If $|\mathcal{O}_z|<u$ then join x to z^1 as an outside singleton in S, blow up W and add a $\mathcal{C}_{H,b}$-factor of $W-x$ to Q. If $|\mathcal{O}_z|=u$ then replace $\mathcal{O}_z$ from $\mathcal{O}$ to $\mathcal{S}$, add a new free odd vertex z^2 to Y_* with $\varphi(z^2)=z$, join W to z^2 and rejoin $\mathcal{O}_z$ from z^1 to z^2.

Case 4b. x is fixed in W and $z\in Y$ is a non-fixed odd vertex. Let R be a forbidden star subgraph of W centered at x such that $W-V(R)$ has a $\mathcal{C}_{H,b}$-factor Q_W. Let $\varphi(z^1)=z$. ($Wz^1\in E(S)$ is possible.) Denote by $\mathcal{O}_z\subseteq\mathcal{O}$ the set of outside stars adjacent to z^1 in S. If $|\mathcal{O}_z|<b$ then join R to z^1 as an outside star in S, blow up W and add Q_W to Q. If $|\mathcal{O}_z|=b$ then replace $\mathcal{O}_z$ from $\mathcal{O}$ to $\mathcal{S}$, add a new fixed odd vertex z^2 to Y_* with $\varphi(z^2)=z$, join W to z^2 and rejoin $\mathcal{O}_z$ from z^1 to z^2.

Case 5. z is contained in a subgraph $W^z\in\mathcal{O}$. Delete W^z from $\mathcal{O}$, blow up W and add a $\mathcal{C}_{H,b}$-factor of $G[V(W)\cup V(W^z)]$ to Q.

Case 6. $z\notin Y\cup\bigcup(\mathcal{S}\cup\mathcal{O})$ is covered by a component P of Q.

- x is non-fixed in W and neither $P\simeq K_2$ and $2\notin H$, nor z is the center of the i-star P with $i+1\notin H$, nor z is a (u,t)-supercenter. In this case $G[V(P)+x]$ has a $\mathcal{C}_{H,b}$-factor Q'. So blow up W, replace P by Q' and add a $\mathcal{C}_{H,b}$-factor of $W-x$ to Q.
- x is non-fixed in W and either $P\simeq K_2$ and $2\notin H$, or z is the center of an i-star with $i+1$ a gap of H. Now $W'=W+xz+P$ is a hedgehog, so grow S by replacing W by W', then delete P from Q.

- x is non-fixed in W and z is either the center of a u-star or a (u,t)-supercenter. Add to Y_* a new free odd vertex z^1 with $\varphi(z^1)=z$, add the leaves of $P-z$ to $\mathcal{S}$ and the forbidden stars of $P-z$ to $\mathcal{O}$ and join z^1 to these new subgraphs and to W in S.
- x is fixed in W and z is not an (s,b)-supercenter. Let R be a forbidden star subgraph of W centered at x such that $W-V(R)$ has a $\mathcal{C}_{H,b}$-factor Q_W. $G[V(P)\cup V(R)]$ has a $\mathcal{C}_{H,b}$-factor Q', so blow up W, delete P from Q and add Q' and Q_W to Q.
- x is fixed in W and z is an (s,b)-supercenter. Add to Y_* a new fixed odd vertex z^1 with $\varphi(z^1)=z$, add the leaves of $P-z$ to $\mathcal{O}$ and the forbidden stars of $P-z$ to $\mathcal{S}$ and join z^1 to these new subgraphs and to W in S.

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